

AN EMBEDDING IN THE FELL TOPOLOGY

ABSTRACT. Using a result from [HZ] it is shown that if a T_2 topological space X contains a closed uncountable discrete subspace, then the space $\omega_1 \times (\omega_1 + 1)$ embeds as a closed subspace of $(CL(X), \tau_F)$, the hyperspace of nonempty closed subsets of X equipped with the Fell topology. We will use the above result to give a partial answer to the Question 3.8 in [CJ].

1. INTRODUCTION

Throughout this paper, let $CL(X)$ denote the family of all nonempty closed subsets of a given T_2 topological space. For $M \in CL(X)$, put

$$M^- = \{A \in CL(X) : A \cap M \neq \emptyset\}, M^+ = \{A \in CL(X) : A \subseteq M\}$$

and denote $M^c = X \setminus M$. The Vietoris topology [Mi] τ_V on $CL(X)$ has as subbase elements of the form U^- , V^+ , where U, V are open in X .

The Fell topology [Fe] τ_F on $CL(X)$ has as a subbase the collection

$$\{U^- : U \text{ open in } X\} \cup \{(K^c)^+ : K \text{ compact in } X\}.$$

It is known [Be] that $(CL(X), \tau_F)$ is Hausdorff (regular, Tychonoff, respectively) iff X is locally compact.

For a metric space (X, d) , let $d(x, A) = \inf\{d(x, a) : a \in A\}$ denote the distance between a point $x \in X$ and a nonempty subset A of (X, d) .

A net $\{A_\alpha : \alpha \in \lambda\}$ in $CL(X)$ is said to be Wijsman convergent to some A in $CL(X)$ if $d(x, A_\alpha) \rightarrow d(x, A)$ for every $x \in X$. The Wijsman topology on $CL(X)$ induced by d , denoted by $\tau_{w(d)}$, is the weakest topology such that for every $x \in X$, the distance functional

$$d(x, \cdot) : CL(X) \rightarrow R^+$$

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is continuous. It can be seen easily that the Wijsman topology on $CL(X)$ induced by d has the family

$$\{U^- : U \text{ open in } X\} \cup \{\{A \in CL(X) : d(x, A) > \epsilon\} : x \in X, \epsilon > 0\}$$

as a subbase [Be].

The above type of convergence was introduced by Wijsman in [Wi] for sequences of closed convex sets in Euclidean space R^n , when he considered optimum properties of the sequential probability ratio test.

2. MAIN RESULT

Claim 2.1. ([HZ]) *Let X be a T_2 topological space which contains an uncountable closed discrete set. Then ω_1 embeds into $(CL(X), \tau_F)$ as a closed set.*

Proof. Let $D = \{x_\alpha : \alpha < \omega_1\}$ be an uncountable closed discrete set in X . Put $E_0 = X$ and

$$E_\alpha = \{x_\eta : \alpha \leq \eta < \omega_1\} \text{ for } \alpha \neq 0, \alpha < \omega_1.$$

Then for every $\alpha < \beta < \omega_1$, $E_\beta \subset E_\alpha$, $E_\alpha = \bigcap_{\beta < \alpha} E_\beta$ for every limit ordinal α and $\bigcap_{\alpha < \omega_1} E_\alpha = \emptyset$.

We will define the mapping $\varphi : \omega_1 \rightarrow (CL(X), \tau_F)$ as follows: $\varphi(\alpha) = E_\alpha$ for every $\alpha < \omega_1$.

We will show that the mapping φ is a homeomorphism between ω_1 and $\{E_\alpha : \alpha < \omega_1\}$. Of course φ is injective.

First we show that φ is continuous. Let $\alpha < \omega_1$ be a limit ordinal. Then $E_\alpha = \bigcap_{\beta < \alpha} E_\beta$. Since $E_\alpha \subset E_\beta$ for every $\beta < \alpha$, it is sufficient to consider $E_\alpha \in (K^c)^+$ for a compact subset $K \subset X$. There must exist $\eta < \alpha$ such that $E_\eta \cap K = \emptyset$, otherwise $E_\alpha \cap K \neq \emptyset$. Thus $\varphi((\eta, \alpha]) \subset (K^c)^+$.

Now we show that φ is open. It is easy to verify that for α isolated, $\varphi(\alpha)$ is isolated in $\varphi(\omega_1)$. Let α be a limit ordinal and consider an open interval $(\beta, \alpha]$. Let $U(x_\alpha)$ be an open neighbourhood of x_α such that $U(x_\alpha) \cap D = \{x_\alpha\}$. Then $U(x_\alpha)^- \cap (\{x_\beta\}^c)^+ \cap \varphi(\omega_1)$ is open in $\varphi(\omega_1)$ and

$$\varphi((\beta, \alpha]) = U(x_\alpha)^- \cap (\{x_\beta\}^c)^+ \cap \varphi(\omega_1).$$

We will show that $\varphi(\omega_1) (= \{E_\alpha : \alpha < \omega_1\})$ is closed in $(CL(X), \tau_F)$. Let $A \in CL(X) \setminus \varphi(\omega_1)$. Suppose first that $A \cap (X \setminus E_1) \neq \emptyset$. Let $x \in X \setminus A$. Put $\mathcal{U} = (X \setminus E_1)^- \cap (\{x\}^c)^+$. Then $A \in \mathcal{U}$ and $\mathcal{U} \cap \varphi(\omega_1) = \emptyset$.

Suppose now that $A \subset E_1$. Put $\alpha_0 = \min\{\beta < \omega_1 : E_\beta \not\supseteq A\}$. Recall $\bigcap_{\alpha < \omega_1} E_\alpha = \emptyset$ and $A \neq \emptyset$. Clearly, α_0 cannot be limit, so $\alpha_0 = \delta + 1$. Thus $x_\delta \in A$ and there must exist $\eta \geq \alpha_0$ such that $x_\eta \notin A$. Let $U(x_\delta)$ be an open neighbourhood of x_δ such that $U(x_\delta) \cap D = \{x_\delta\}$. Put $\mathcal{U} = U(x_\delta)^- \cap (\{x_\eta\}^c)^+$. Then $A \in \mathcal{U}$ and $\mathcal{U} \cap \varphi(\omega_1) = \emptyset$. \square

Claim 2.2. *Let X be a T_2 topological space which contains an uncountable closed discrete set. Then $\omega_1 + 1$ embeds into $(CL(X), \tau_F)$ as a closed set.*

Proof. Let $D = \{x_\alpha : \alpha < \omega_1\}$ be an uncountable closed discrete set in X as above and also let $E_\alpha, \alpha < \omega_1$ be sets defined in the proof of Claim 1. For every $\alpha \leq \omega_1$, define the sets F_α as follows:

$$F_\alpha = E_\alpha \cup \{x_0\}, \alpha < \omega_1 \text{ and } F_{\omega_1} = \{x_0\}.$$

Define the mapping $\eta : \omega_1 + 1 \rightarrow (CL(X), \tau_F)$ as follows: $\eta(\alpha) = F_\alpha, \alpha \leq \omega_1$.

Using the proof of the Claim 2.1 it is easy to verify that η embeds $\omega_1 + 1$ into $(CL(X), \tau_F)$ as a closed subspace. \square

Theorem 2.1. *Let X be a T_2 topological space which contains an uncountable closed discrete set. Then $\omega_1 \times (\omega_1 + 1)$ embeds into $(CL(X), \tau_F)$ as a closed set.*

Proof. Let D be an uncountable closed discrete set in X . We express D as the disjoint union

$$D = D_0 \cup D_1,$$

such that $|D_0| = \aleph_1 = |D_1|$. We enumerate $D_0 = \{x_\alpha : \alpha < \omega_1\}$ and $D_1 = \{y_\alpha : \alpha < \omega_1\}$ and put $E_0 = X, E_\alpha = \{x_\beta : \alpha \leq \beta < \omega_1\}, \alpha \neq 0, \alpha < \omega_1, F_0 = X, F_\alpha = \{y_\beta : \alpha \leq \beta < \omega_1\} \cup \{y_0\}, \alpha \neq 0, \alpha < \omega_1$ and $F_{\omega_1} = \{y_0\}$.

Define now the mappings φ and η as in the above Claims:

$$\varphi : \omega_1 \rightarrow (CL(X), \tau_F), \varphi(\alpha) = E_\alpha, \alpha < \omega_1,$$

$$\eta : \omega_1 + 1 \rightarrow (CL(X), \tau_F), \eta(\alpha) = F_\alpha, \alpha \leq \omega_1$$

and define $\Pi : \omega_1 \times (\omega_1 + 1) \rightarrow (CL(X), \tau_F)$ as follows: $\Pi(0, 0) = X, \Pi(0, \beta) = \eta(\beta), \beta \neq 0, \beta \leq \omega_1, \Pi(\alpha, 0) = \varphi(\alpha), \alpha \neq 0, \alpha < \omega_1$ and

$$\Pi(\alpha, \beta) = \varphi(\alpha) \cup \eta(\beta), \alpha \neq 0 \neq \beta, \alpha < \omega_1, \beta \leq \omega_1.$$

Put $\mathcal{L} = \Pi(\omega_1 \times (\omega_1 + 1))$. We claim that Π is an embedding from $\omega_1 \times (\omega_1 + 1)$ onto (\mathcal{L}, τ_F) and \mathcal{L} is a closed subspace of $(CL(X), \tau_F)$.

Clearly, Π is one-to-one. Using the continuity of $\varphi : \omega_1 \rightarrow (\varphi(\omega_1), \tau_F)$ and the continuity of $\eta : \omega_1 + 1 \rightarrow (\eta(\omega_1 + 1), \tau_F)$ it is easy to verify that also $\Pi : \omega_1 \times (\omega_1 + 1) \rightarrow (\mathcal{L}, \tau_F)$ is continuous. Also using the openness of $\varphi : \omega_1 \rightarrow (\varphi(\omega_1), \tau_F)$ and the openness of $\eta : \omega_1 + 1 \rightarrow (\eta(\omega_1 + 1), \tau_F)$ it is easy to see that $\Pi : \omega_1 \times (\omega_1 + 1) \rightarrow (\mathcal{L}, \tau_F)$ is open.

We will show that \mathcal{L} is closed in $(CL(X), \tau_F)$. Let $A \in CL(X) \setminus \mathcal{L}$. Suppose first that $A \cap (X \setminus (E_1 \cup D_1)) \neq \emptyset$. Let $x \in X \setminus A$. Put

$$\mathcal{V} = (X \setminus (E_1 \cup D_1))^- \cap (\{x\}^c)^+.$$

Then $A \in \mathcal{V}$ and $\mathcal{V} \cap \mathcal{L} = \emptyset$.

Suppose now that $A \subset E_1 \cup D_1$. First, suppose that $A \subset E_1$. We use the proof of Claim 2.1, where for $\mathcal{U} = U(x_\delta)^- \cap (\{x_\eta\}^c)^+$, $\delta < \eta$, we have $A \in \mathcal{U}$ and $\mathcal{U} \cap \varphi(\omega_1) = \emptyset$. Without loss of generality we can suppose that $U(x_\delta)$ is an open neighbourhood of x_δ such that $U(x_\delta) \cap (D_0 \cup D_1) = \{x_\delta\}$. We claim that $\mathcal{U} \cap \mathcal{L} = \emptyset$. Suppose there is some (α, β) with $\Pi((\alpha, \beta)) \in \mathcal{U}$, then $\varphi(\alpha) \in U(x_\delta)^- \cap (\{x_\eta\}^c)^+$, a contradiction. Similarly, if $A \subset D_1$.

Suppose now, that $A \cap E_1 \neq \emptyset$ and $A \cap D_1 \neq \emptyset$. Then either $A \cap E_1 \notin \varphi(\omega_1)$ or $A \cap D_1 \notin \eta(\omega_1 + 1)$. So we can again use the above idea. \square

3. EMBEDDING IN THE WIJSMAN TOPOLOGY

In this part we will give a partial answer to the Question 3.8 in [CJ].

Question 3.8 [CJ] Let (X, d) be a metric space. If $(CL(X), \tau_{w(d)})$ is non-normal, does $(CL(X), \tau_{w(d)})$ contain a closed copy of $\omega_1 \times (\omega_1 + 1)$?

It is known [Be] that if a metric space (X, d) has nice closed balls, then the Fell topology τ_F and the Wijsman topology $\tau_{w(d)}$ on $CL(X)$ coincide.

A metric space (X, d) is said to have nice closed balls [Be] provided whenever B is a closed ball in X that is a proper subset of X , then B is compact.

It is also known [LL] that if (X, d) is a metric space, then $(CL(X), \tau_{w(d)})$ is metrizable if and only if (X, d) is separable.

Theorem 3.1. *Let (X, d) be a metric space with nice closed balls. If $(CL(X), \tau_{w(d)})$ is non-normal, then $(CL(X), \tau_{w(d)})$ contains a closed copy of $\omega_1 \times (\omega_1 + 1)$.*

Moreover we have the following theorem which gives a better partial answer to the Question 3.8 in [CJ].

Theorem 3.2. *Let (X, d) be a metric space such that every closed proper ball in X is totally bounded. If (X, d) is non-separable, then $(CL(X), \tau_{w(d)})$ contains a closed copy of $\omega_1 \times (\omega_1 + 1)$.*

Proof. Since (X, d) is non-separable, there exist $\epsilon > 0$ and a set $D \subset X$ with $|D| = \aleph_1$ which is ϵ -discrete, that is, $d(x, y) \geq \epsilon$ for all distinct $x, y \in D$. We express D as a disjoint union $D = D_0 \cup D_1$ as in the proof of Theorem 2.1 and we will proceed as in the proof of Theorem 2.1. We claim that $\Pi : \omega_1 \times (\omega_1 + 1) \rightarrow (CL(X), \tau_{w(d)})$ is embedding. Of course, it is sufficient to prove that Π is continuous.

It is sufficient to verify that if $d(x, \Pi((\alpha_0, \beta_0))) > r$ for some $x \in X$ and $r > 0$, then $d(x, \Pi((\alpha, \beta))) > r$ for all (α, β) from a neighbourhood of (α_0, β_0) . However, it is clear, since the closed proper ball with center x and the radius r can contain only finitely many points of the set D . □

Theorem 3.2 gives another proof of the known result (Corollary 5.8, [HN]), which claims, that if (X, d) is a metric space in which every closed proper ball is totally bounded, then $(CL(X), \tau_{w(d)})$ is normal if and only if $(CL(X), \tau_{w(d)})$ is metrizable.

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